Indian Statistical Institute, Bangalore Centre.
End-Semester Exam : Stochastic Processes (B3)

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Max. points : 40. **Time Limit : 2.5 hours.**

There are two parts to the question paper - PART A and PART B. Read the instructions for each section carefully. Some common definitions and inequalities are listed on Page 3.

1 PART A : MULTIPLE-CHOICE QUESTIONS - 10 Points.

Please write only the correct choice(s) (for ex., (a), (b) et al.) in your answer scripts. No explanations are needed. Write PART A answers in a separate page.

Some questions will have multiple correct choices. Answer all questions. Each question carries 2 points. 1 point will be awarded if only some correct choices are chosen and no wrong choices are chosen.

1. Let $X_1, \ldots, X_n$ be identically distributed bounded random variables and $S_n = \sum_{i=1}^n X_i$. Which of the following are true for all $n \geq 1$?
   (a) $\mathbb{E}[S_n] = n\mathbb{E}[X_1]$.
   (b) $\mathsf{VAR}(S_n) = n\mathsf{VAR}(X_1)$.
   (c) $\mathsf{VAR}(S_n) \leq n^2\mathsf{VAR}(X_1)$.
2. Let $X_1, \ldots, X_n$ be i.i.d. symmetric Bernoulli (∓ 1) random variables and $S_n = \sum_{i=1}^n X_i$. Which of the following are true for all $n \geq 1$?
   (a) $\mathbb{P}\{|S_n| \geq 4\sqrt{n}\} \leq \frac{1}{16}$.
   (b) $\mathbb{E}[|S_n|] \leq \sqrt{n}$.
   (c) $\mathbb{P}\{S_n \geq 0\} < \frac{1}{2}$.
3. Consider simple random walk on the complete graph K_n on n vertices for $n \geq 3$. Which of the following is true for all $n \geq 3$?
   (a) It is irreducible.
   (b) It is periodic.
   (c) $P(x, y) > 0$ for all $x \neq y \in K_n$?
4. Let P be a transition matrix on $\{0, 1\}$ with $p = P(0, 1)$ and $q = P(1, 0)$. Assume that $p, q \in (0, 1)$. Which of the following are true for all $p, q \in (0, 1)$?
   (a) P is irreducible.
   (b) P is reversible.
   (c) P is periodic.
5. Let P be a transition matrix reversible w.r.t. probability distribution π. Which of the following are true ?

(a) π is a stationary distribution for P.

(b) $\frac{P+I}{2}$ is reversible w.r.t. π.

(c) P^2 is not a reversible transition matrix.

2 PART B : 30 Points.

Answer any three questions only. All questions carry 10 points.

Give necessary justifications and explanations for all your arguments. If you are citing results from the class, mention it clearly. Always define the underlying random variables and events clearly before computing anything !

1. (a) For a random variable X, define $\|X\|_G := \inf\{t > 0 : \mathbb{E}\Big[e^{X^2/t^2}\Big] \leq 2\}$. Show that $\|.\|_G$ defines a norm on the space of sub-Gaussian random variables. **(5)**

   (b) If X, Y are sub-Gaussian, show that XY is sub-exponential. **(5)**

2. Let X_t be an irreducible HMC on V, a countable set. Fix $A \subset V$ and $H_A = \inf\{t \geq 0 : X_t \in A\}$ with $\inf \emptyset = \infty$. Set $h_i := \mathbb{P}_i[H_A < \infty]$. Show that $h_i, i \geq 1$ is the minimal (non-negative) solution to the following set of linear equations:

$$x_i = 1,\, i \in A \;;\; x_i = \sum_j P(i,j)x_j,\, i \notin A.$$

   Here minimality means that if x_i are non-negative solutions to the above linear equations then $x_i \geq h_i$ for all i.

3. (a) Consider the square lattice $G = \mathbb{Z}^2$ with unit weights on edges. Let $G_n, n \geq 1$ be an exhausting sequence obtained by the restriction of the graph G to $V_n = [-n,n]^2$. Let $Z_n = V_n^c$ and $\mathcal{R}_n := \mathcal{R}(0 \leftrightarrow Z_n; G_n)$ denote effective resistance between 0 and Z_n in G_n. Show that $\liminf_{n\to\infty}(\log n)^{-1}\mathcal{R}_n > 0$. **(5)**

   (b) Let X_t be a stochastic process taking values in V, adapted to a filtration $\mathcal{F}_t$ and let P be a stochastic matrix with Δ being the corresponding Laplacian operator. Assume that for any bounded function $f : V \to \mathbb{R}$, the process $M_t^f := f(X_t) - \sum_{s=0}^{t-1} \Delta f(X_s)$ is a $\mathcal{F}_t$-martingale. Show that $X_t, t \in \mathbb{Z}_+$ is a HMC. **(5)**

4. Let $G_n = G(n,p)$ be the Erdős-Rényi random graph on n vertices with edge-probability p. Let $Cl(G_n)$ denote the size (in terms of vertices) of the largest clique in G_n. Show that

$$\lim_{n\to\infty} \mathbb{P}\left\{Cl(G_n) \geq 4\right\} = \begin{cases} 0 & \text{if } n^{2/3}p \to 0. \\ 1 & \text{if } n^{2/3}p \to \infty. \end{cases}$$

5. (a) Let Q be a symmetric and irreducible transition matrix on a finite vertex set V. Let π be a strictly positive probability distribution on V. Define a new matrix P as follows:

$$P(x,y) = Q(x,y)\left[\frac{\pi(y)}{\pi(x)} \wedge 1\right],$$

   with $P(x,x) = 1 - \sum_{z:z\neq x} P(x,z)$. Show that P is an irreducible transition matrix and reversible. Find the stationary distribution. **(5)**

   (b) Let $A = (a_{i,j})$ be a symmetric matrix. Assume that for all $x_i \in \{-1,+1\}$, $|\sum_{i,j} a_{i,j}x_ix_j| \leq 1$. Show that there exists an universal finite constant K (not depending on A) such that for all $N \geq 1$, and $u_i, v_j \in \mathbb{R}^N$ with $\|u_i\| = \|v_j\| = 1$, it holds that

$$|\sum_{i,j} a_{i,j}\langle u_i, v_j\rangle| \leq 2K. \qquad \textbf{(5)}$$

Sub-Gaussian random variable with variance factor ν: $\Psi_{X-\mu}(s) \le s^2\nu/2$, for $s \in \mathbb{R}$ and where Ψ is the cumulant generating function i.e., $\Psi_X(s) = \log \mathbb{E}\left[e^{sX}\right]$.

Space of sub-Gaussian random variables: This is set $\{X : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R} : X \text{ is sub-Gaussian}\}$ with the equivalence relation $X \sim Y$ if $X = Y$ a.s. and $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

Sub-exponential random variable with parameters ν and α: $\Psi_{X-\mu}(s) \le s^2\nu/2$, for $|s| \le \alpha^{-1}$. Equivalently, X is sub-exponential if there exists $K \in [0, \infty)$ such that $\Psi_{|X|}(s) \le Ks$ for $0 \le s \le K^{-1}$.

Hölder's inequality: Let $w_j \ge 0$, $\sum_{j=1}^n w_j \le 1$ be weights and Y_j's be non-negative random variables such that $\mathbb{E}\left[Y_j^{1/w_j}\right] < \infty$ for all j. Then, we have that

$$\mathbb{E}\left[\prod_{j=1}^n Y_j\right] \le \prod_{j=1}^n \mathbb{E}\left[Y_j^{1/w_j}\right]^{w_j}.$$

Gamma function: $\Gamma(a) = \int_0^\infty e^{-s} s^{a-1}\, ds$ for $a > 0$ and $\Gamma(a) \le a^a$.

Stirling's Approximation: $\sqrt{2\pi n}\ (\frac{n}{e})^n\ e^{1/(12n+1)} \le n! \le \sqrt{2\pi n}\ (\frac{n}{e})^n\ e^{1/(12n)}$.

Laplacian: For a bounded $f : V \to \mathbb{R}$ and stochastic matrix P, define the Laplacian operator as $\Delta f(x) := \sum_y P(x,y)f(y) - f(x)$.

Erdős-Rényi random graph $G(n,p)$: This is the random sub-graph of the complete graph where every edge is retained with probability p and independently of each other.

Clique: Clique is a complete sub-graph. For example, a k-clique is a complete graph on k vertices and size of such a clique is k.